Dispersion Formulæ and the Polarisation of Scattered Light: with Application to Hydrogen.

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Introduction and Summary.

Since Lord Rayleigh's experimental study of the polarisation of light scattered by gases, the subject has been examined theoretically from various points of view; in particular by the late Lord Rayleigh,[*] M. Born,[†] Sir J. J. Thomson,[‡] and R. Gans.[§] The object of the following paper is to connect the polarisation of light scattered by a simple homogeneous medium with the constants of the dispersion formula for that medium. The result of the calculation gives the ratio of the intensities of the two polarised components for all wave-lengths for which the dispersion formula is valid. Although the expressions can be generalised, they are given in the first instance in their simplest form.

Assuming the medium to be a gas whose approximate dispersion formula contains only one natural frequency in the ultra-violet, this is generalised by supposing the molecules to be anisotropic; the single natural frequency is then replaced by a triplet, or by a doublet for molecules with axial symmetry. In the latter case the dispersion formula is, in the usual notation

$$n^2-1 = \tfrac{1}{3}C\{(p_1^2-p^2)^{-1}+2(p_2^2-p^2)^{-1}\}.$$

The three constants of the formula are left to be determined from experimental results. For this purpose it is necessary to have accurate values of the refractive index over a large range of wave-lengths, and hydrogen has been chosen as the most suitable. Two formulæ of this type are obtained, one of prolate form and the other oblate; both express the dispersion with sufficient accuracy over the range 5462 to 1854 Ångström units. Incidentally a comparison is made with previous formulæ such as those of Cuthbertson, Lundblad, Debye and Sommerfeld.

The expressions for the ratio of the intensities of the two polarised components of the scattered light are based directly upon Rayleigh's calculations; an obvious interpretation of the coefficients brings them into relation with the particular type of dispersion formula. In the special case given above, if the formula is written as

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[*] Rayleigh, 'Phil. Mag.,' vol. 35, p. 373 (1918); or 'Scientific Papers,' vol. 6, p. 540.

[†] M. Born, 'Deutsch. Phys. Gesell.,' vol. 19, p. 243 (1917), and vol. 20, p. 16 (1918).

[‡] Sir J. J. Thomson, 'Phil. Mag.,' vol. 40, p. 393 and p. 713 (1920).

[§] R. Gans, 'Ann. der Phys.,' vol. 65, p. 97 (1921).

$$n^2-1 = \tfrac{1}{3}\{n_1^2-1+2(n_2^2-1)\},$$

the corresponding ratio of intensities when the primary beam is unpolarised is

$$6(n_1^2-n_2^2)^2/\{45(n^2-1)^2+7(n_1^2-n_2^2)^2\}.$$

The values have been calculated, at various wave-lengths, for hydrogen from the two dispersion formulæ; the numerical results are in substantial agreement with the revised experimental value 0·0383 obtained by Lord Rayleigh. Other dispersion formulæ are also examined from this point of view. It is shown that their failure to give the correct dispersion in the extreme ultra-violet is accompanied by a corresponding divergence in the ratio of the intensities of the components of scattered light.

It is hoped to extend the investigation to a comparative study of gases or other media for which the available experimental results are suitable for analysis by this method.

Dispersion with Anisotropic Molecules.

In deriving the simplest type of dispersion formula,

$$n^2-1 = C/(p_0^2-p^2), \tag{1}$$

it is assumed that the molecules are isotropic, the electric polarisation being always in the direction of the electric force. The formula (1) has been generalised in various ways; for instance, by direct summation of terms on the right for different natural frequencies, or by more precise specification of the local electric field leading to similar formulæ with $(n^2-1)/(n^2+2)$ or more general forms on the left, or again by using some definite theory of atomic or molecular structure. We shall leave these out of account and consider a simple gaseous medium composed of anisotropic molecules with random distribution and orientation. Assume that each molecule has three principal axes of polarisation fixed relative to the molecule and that the connection between the polarisation (P_1, P_2, P_3) and the electric field (F_1, F_2, F_3) is given by three equations of the form

$$\frac{\partial^2 P_1}{\partial t^2}+p_1^2 P_1 = kF. \tag{2}$$

It should be noted that for simplicity we assume the coefficient k to be the same for the three components. Let $O(x, y, z)$ be axes fixed in space, and consider a plane wave travelling in the direction Ox with the electric force Z parallel to Oz and proportional to e^{ipt}. The directions of the principal axes of any molecule may be specified by two angular co-ordinates θ, ϕ; and we obtain F_1, F_2 and F_3 by resolving Z along the principal axes. Further, the mean polarisation is obtained by summation for all possible orientations of the molecules, assuming random distribution with respect to θ and ϕ. Finally, if P is the component of mean polarisation parallel to Oz, the dispersion formula is derived from the relation $P/Z = (n^2-1)/4\pi$. The process gives readily the formula

$$n^2-1 = \tfrac{1}{3}C\left(\frac{1}{p_1^2-p^2}+\frac{1}{p_2^2-p^2}+\frac{1}{p_3^2-p^2}\right), \tag{3}$$

where C has been written for $4\pi Nk$, with N as the number of molecules per unit volume.

In Drude's interpretation of (1) in terms of vibrating electrons, the value of C is $4\pi Ne^2/m$ or some integral multiple thereof; and the same holds for the formula (3). It is well known that the value of e/m derived from dispersion formulæ is less than the accepted value for an electron. This may be ascribed, no doubt, to unknown coupling effects between vibrating electrons, and it might be supposed that the generalisation from (1) to (3) might remove this discrepancy in part; however, we shall find it better not to make any assumption in this respect, but to leave C to be determined from the experimental values of the dispersion.

It is convenient to write (3) in the form

$$n^2-1 = \tfrac{1}{3}(n_1^2-1+n_2^2-1+n_3^2-1), \tag{4}$$

where $n_1^2-1 = C/(p_1^2-p^2)$; etc.

The quantities n_1, n_2, n_3 are subsidiary principal indices; in fact, if the molecules were all orientated in the same way, without thereby introducing any effects due to mutual interference, the medium would be optically a crystal with these principal indices.

If the molecules have an axis of symmetry, so that $p_2 = p_3$, the formulæ become

$$n^2-1 = \tfrac{1}{3}C\left(\frac{1}{p_1^2-p^2}+\frac{2}{p_2^2-p^2}\right), \tag{5}$$

$$= \tfrac{1}{3}\{(n_1^2-1)+2(n_2^2-1)\}. \tag{6}$$

The formula (5) is to be calculated from experimental results as a three-constant formula. If it does not prove adequate it may be necessary to use (3) or its more general forms, or, further, to introduce into (3) and (5) some of the extensions which were mentioned in connection with formula (1).

It may be noted that, instead of, as in (1), a single resonance period at which n becomes infinite, we have in (3) a triplet and in (5) a doublet; but the positions of maximum absorption or reflection cannot be discussed satisfactorily without introducing terms of some kind to represent dissipation.

Scattering by Anisotropic Molecules.

There are several factors in Rayleigh's well-known expression for the intensity of light scattered by a medium composed of a random distribution of small particles or molecules, each with spherical symmetry. Each particle behaves like a vibrating doublet, and hence there is a factor λ^{-4}, where λ is the wave-length; the factor with which we are concerned here is that which expresses the optical properties of the particles, namely $(n^2-1)^2$, where n is the refractive index of the medium.

In the paper referred to above,[*] an extension was made to scattering by a cloud of similar small particles of any shape and oriented at random. We quote Rayleigh's results for the ratio of intensities of the two polarised components in the light scattered at right angles, the principal component being that which vibrates parallel to the primary vibrations, and the subordinate component (vanishing for spherical particles) being that in which the vibrations are perpendicular to the primary vibrations. The scattering effect of a particle, or molecule, is represented by coefficients; taking first the case of particles with an axis of symmetry, these coefficients are written A, A, C. Then by a simple, but general, argument, expressions are found for the aggregate intensities due to a large number of particles with their positions and their axes distributed at random. The results are: (i) if the primary light is plane polarised, the ratio of the intensities of the two components is

$$(C-A)^2/(8A^2+3C^2+4AC), \tag{7}$$

and (ii) if the primary light is unpolarised, the ratio is

$$2(C-A)^2/(9A^2+4C^2+2AC). \tag{8}$$

For the more general case, when each particle has three principal axes, the coefficients being A, B, C, the corresponding results are in case (i)

$$\frac{A^2+B^2+C^2-BC-CA-AB}{3(A^2+B^2+C^2)+2(BC+CA+AB)}, \tag{9}$$

and in case (ii)

$$\frac{2(A^2+B^2+C^2-BC-CA-AB)}{4(A^2+B^2+C^2)+BC+CA+AB}. \tag{10}$$

The paper concludes with the remark that "the problem of an ellipsoidal particle of uniform dielectric quality can be no more than illustrative of what happens in the case of a molecule; but we may anticipate that the general form, with suitable values of A, B, C, still applies, except it may be under special circumstances, where resonance occurs, and where the effective values of the coefficients may vary greatly with the wave-length of the light."

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[*] Rayleigh, 'Scientific Papers,' vol. 6, p. 540.

The interpretation of the coefficients which we make now is obvious when we remember their values for spherical particles when A = B = C, and we take into account the corresponding dispersion formulæ (4), (5) and (6); in the notation introduced there, we take

$$A : B : C = n_1^2-1 : n_2^2-1 : n_3^2-1. \qquad (11)$$

Substituting these values in (7), (8), (9), (10) we obtain the required forms. It is convenient, instead of leaving them entirely in terms of the sub-indices n_1, n_2, n_3, to introduce the total refractive index n, in accordance with (4) and (6). Writing the results in the same order as in (7)–(10), the ratios in the four cases are

$$\frac{3\,(n_1^2-n_2^2)^2}{45\,(n^2-1)^2+4\,(n_1^2-n_2^2)^2}, \qquad (12)$$

$$\frac{6\,(n_1^2-n_2^2)^2}{45\,(n^2-1)^2+7\,(n_1^2-n_2^2)^2}, \qquad (13)$$

$$\frac{3\,\{(n_2^2-n_3^2)^2+(n_3^2-n_1^2)^2+(n_1^2-n_2^2)^2\}}{90\,(n^2-1)^2+4\,\{(n_2^2-n_3^2)^2+(n_3^2-n_1^2)^2+(n_1^2-n_2^2)^2\}}, \qquad (14)$$

$$\frac{6\,\{(n_2^2-n_3^2)^2+(n_3^2-n_1^2)^2+(n_1^2-n_2^2)^2\}}{90\,(n^2-1)^2+7\,\{(n_2^2-n_3^2)^2+(n_3^2-n_1^2)^2+(n_1^2-n_2^2)^2\}}. \qquad (15)$$

The first term in the denominator in each case is the same as if the molecules were isotropic; it may be said to represent the effect of random distribution as regards position. The anisotropy of the molecules, and the random distribution of their axes as regards orientation, contribute to the intensities of both polarised components, and are represented by the numerator and the second term in the denominator in each case.

These expressions are obviously associated with dispersion formulæ of types (4) and (5) with n^2-1 expressed as a function of the wave-length and proportional to density; it may be anticipated that they will have a similar range of application. They are capable of generalisation in a similar manner to the dispersion formulæ. They could also be extended in accordance with other methods of obtaining the intensity of scattered light, for instance, Einstein's more general method for isotropic molecules, based on a statistical view of the fluctuations of density. The latter idea has been applied to a gas with anisotropic molecules by Born in the paper already quoted. Born distinguishes between scattering due to fluctuations of density and that due to variations in orientation; adding the two, his results for the total scattering are given in general form in terms of functions of the wave-length.

The special case (13) which we shall use—which we have derived directly from Rayleigh's expressions—can, in fact, be deduced from Born's formulæ when they are interpreted from the present point of view.

Dispersion Formulæ for Hydrogen.

The most recent determinations of the dispersion of hydrogen are those of M. Kirn,* over wave-lengths ranging from 5462 to 1854 Ångström units. His values agree well with those of Koch, whose shortest wave-length was 2302, and with those of Cuthbertson in the visible spectrum. We shall use Kirn's results, and the wave-lengths and values of $(n-1).10^7$ are given in the first and second columns of Table I.

We try now to fit these with a formula

$$n^2-1 = \mathrm{C}\left(\frac{1}{\lambda_1^{-2}-\lambda^{-2}}+\frac{2}{\lambda_2^{-2}-\lambda^{-2}}\right). \qquad (16)$$

Most dispersion formulæ derived from experimental results are expressed in Cauchy's form, a power series in λ^2; in that case the formula can be made to have any required degree of accuracy, or, with a limited number of terms of the series, the coefficients for the best possible agreement can be found by the method of least squares. Such methods, adapted to the form (16), would become very laborious. It was decided not to attempt adjusting the constants, but simply to determine them from three pairs of values of n and λ; the values of λ chosen were the two extremes 5462 and 1854, together with 2753.

We have then three equations of the type

$$\alpha' = \mathrm{C}\left(\frac{1}{x_1-x'}+\frac{2}{x_2-x'}\right), \qquad (17)$$

from which to determine C, x_1, and x_2. By elimination an equation of the fifth degree in x_1 is obtained, and a similar equation for x_2. The numerical coefficients were calculated so as to allow the roots of these equations to be found to four significant figures. Taking a pair of values so found, successive approximation was made until the three values of C from equations (17) were the same to a sufficient degree of accuracy. In this way we obtain the formula, with λ in centimetres,

$$n^2-1 = 1{\cdot}2429583\times10^{-4}\left\{\frac{1}{1{\cdot}032551-10^{-10}\lambda^{-2}}+\frac{2}{1{\cdot}638331-10^{-10}\lambda^{-2}}\right\}. \qquad (18)$$

The values of n were then calculated from (18) over the whole range, and the differences between the calculated and observed values of $(n-1).10^7$ are shown in Column 3 of Table I. The differences are within the possible experimental errors, and the agreement is sufficiently good; especially taking into account that equal weight has been given to the value in the extreme ultra-violet where the possible experimental error is greatest, and that the constants have not been adjusted to give the best average agreement. Returning to the equations in x_1 and x_2 derived from (17) there are other possible roots. We find the equation in x_2 has a root near 1·169; further, it is seen that in order to make a possible pair there should be a root in x_1 about the value 2. In this case the equation in x_1 has no actual root, but it was found by graphical methods that a slight alteration in the values of α in (17) would have given such a root; and, further, that the corresponding value of C would have been practically the same as in (18). We therefore assume the same value of C, and determine x_1 and x_2 from the two equations of (17) corresponding to the two wave-lengths 5462 and 1854; it will be seen that this also very nearly agrees with the equation for the intermediate wave-length 2753. We have now a second dispersion formula

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* M. Kirn, 'Ann. der Phys.,' vol. 64, p. 566 (1921).

$$n^2-1 = 1\cdot2429583\times10^{-4}\left\{\frac{1}{2\cdot088392-10^{-10}\lambda^{-2}}+\frac{2}{1\cdot169513-10^{-10}\lambda^{-2}}\right\}. \tag{19}$$

The differences between the values of $(n-1).10^7$ calculated from (19) and the observed values are shown in Column 4 of Table I. It will be seen that there is the same degree of accuracy as with formula (18).

It is perhaps unnecessary to remark that in numerical calculation of formulæ of this type with three constants at our disposal, a certain amount of variation of the values of the constants is possible, without making any great difference to the agreement between observed and calculated values of the dispersion. It has therefore been thought desirable to state explicitly the process adopted in obtaining (18) and (19). This point may be illustrated by an empirical formula given by Kirn to express his experimental results, without stating how the constants were determined, namely,

$$(n-1)\,.\,10^8 = \frac{1220070}{118\cdot6371-10^{-8}\lambda^{-2}}+\frac{581198\cdot1}{175\cdot3273-10^{-8}\lambda^{-2}}. \tag{20}$$

The differences between observed values and those calculated from (20) are shown in Column 5 of Table I. It is of interest to notice that this empirical formula is nearly of the type we require, with one numerator twice the other.

Returning to (18) and (19) we have two dispersion formulæ for hydrogen, one being of the prolate and the other of the oblate type. This, of course, has no reference to actual shape of molecules; it is using the words in the sense in which they are used in describing uniaxial crystals, such as hydrogen might be on this view, with the molecules all oriented the same way. It cannot be said that the available data enable us to decide between the alternatives. If the constant C in (18) and (19) agreed with Drude's inter-

Table I.—Differences between Calculated and Observed Values of $(n-1) \cdot 10^7$ for Hydrogen.

λ.	$(n-1) \cdot 10^7$ (obs.).	(18).	(19).	Kirn.	Cuthbertson.	Lundblad.	Debye.
5462·260	1396·50	0·00	0·00	−0·26	+ 0·90	+0·07	−26·0
4359·567	1417·73	+0·22	+0·31	−0·07			
4078·991	1426·32	+0·23	+0·37	−0·01			
4047·698	1427·41	+0·21	+0·35	+0·01			
3342·438	1461·33	+0·10	+0·36	+0·25	− 3·24	−0·16	−28·0
2968·141	1491·01	+0·25	+0·60	+0·40			
2894·452	1498·59	+0·15	+0·51	+0·42	—	—	−36·0
2753·615	1515·00	0·00	+0·41	+0·48	− 7·37	−0·15	
2535·560	1546·90	−0·12	+0·35	+0·57			
2379·115	1576·81	−0·43	+0·07	+0·61			
2302·870	1594·18	−0·58	−0·08	+0·60	—	+0·49	
1990·469	1693·95	−0·64	−0·46	+0·25			
1935·846	1718·24	−0·46	−0·22	+0·07	−27·63	+5·41	
1862·749	1755·41	−0·01	+0·01	−0·20			
1854·637	1759·96	0·00	0·00	+0·78	−32·25	+8·34	−58·0

pretation, its value, for two electrons per molecule, would be 1·6059; the smaller value 1·24296 is required for a simple equivalent molecule, with anisotropic properties of polarisation, without specifying details of structure.

Calculating the value of n^2 when $\lambda = \infty$, we obtain 1·000272 from both (18) and (19). This is the value of the dielectric constant given by Tangl,* who also remarks that it agrees with the value of n^2_∞ derived from dispersion formulæ of Cauchy's type. This may be taken to confirm the view that the hydrogen molecule contains no permanent doublets, such as have been invoked to explain the variation of dielectric constants with temperature in certain cases.†

Finally, for the wave-lengths at which n becomes infinite, we have from (18) the doublet

$$\lambda_1 = 984{\cdot}07\,; \qquad \lambda_2 = 781{\cdot}3\,; \tag{21}$$

and from (19) the doublet

$$\lambda_1 = 699{\cdot}2\,; \qquad \lambda_2 = 924{\cdot}7. \tag{22}$$

It is of interest to compare these formulæ with others which have been given for hydrogen.

The simple formula due to Cuthbertson‡ was based on observations over a comparatively short range; expressed in terms of wave-length instead of frequency, it becomes

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* K. Tangl, 'Ann. der Phys.,' vol. 26, p. 59 (1908).

† H. Riegger, 'Ann. der Phys.,' vol. 59, p. 753 (1919).

‡ C. and M. Cuthbertson, 'Roy. Soc. Proc.,' A, vol. 83, p. 165 (1910).

$$n^2-1=\frac{3\cdot76\times10^{-4}}{1\cdot378777-10^{-10}\lambda^{-2}}. \tag{23}$$

This means a single dominant wave-length $\lambda_0 = 851\cdot6$. Comparing with (21) and (22) we see that λ_0 is a mean value for the lines of the doublet in each case, being roughly equal to $\frac{1}{3}(\lambda_1+2\lambda_2)$. A few values calculated from (23) are given in Column 6 of Table I, to show how the differences increase rapidly in the ultra-violet although there may be good agreement over the small range from which the formula was calculated.

In a more elaborate formula due to Lundblad,* the hydrogen molecule is supposed to contain two revolving electrons at a distance r apart. The formula is given as

$$\frac{n^2-1}{n^2+2}=\frac{4\pi N}{3}\cdot\frac{1}{1-\xi-2\xi^2}\cdot\frac{2}{H}, \tag{24}$$

where $\quad H = m(p_0^2-p^2)/e^2; \qquad \xi = 1/r^3H.$

Lundblad uses the accepted electronic value of e/m, and then determines the two remaining constants from Koch's values of the dispersion; his results are

$$p_0^2 = 8\cdot20892\times10^{32}; \qquad r = 1\cdot021\times10^{-8}\text{ cm.}$$

Now (24) is in fact of the type we have considered in (5); for, putting the right-hand side into partial fractions, we obtain

$$\frac{n^2-1}{n^2+2}=\frac{8\pi Ne^2}{9m}\left\{\frac{1}{p_0^2-2p_1^2-p^2}+\frac{2}{p_0^2+p_1^2-p^2}\right\}, \tag{25}$$

where $mp_1^2=e^2/r^3$. Using the numerical values given above, and expressing in wave-lengths for comparison with (18) and (19), Lundblad's formula becomes

$$\frac{n^2-1}{n^2+2}=\frac{1\cdot605896\times10^{-4}}{3}\left\{\frac{1}{0\cdot976725-10^{-10}\lambda^{-2}}+\frac{2}{2\cdot977210-10^{-10}\lambda^{-2}}\right\}. \tag{26}$$

The calculated values for some wave-lengths are shown in Column 7 of Table I. Although there is good agreement over the range to which Koch's experiments extended, the values become increasingly too large in the extreme ultra-violet. This may be connected with the electronic value of the constant C being assumed in advance, and so leading to too large a separation of the lines of the doublet in (26) compared with those in (18) or (19).

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* R. Lundblad, 'Untersuch. über die Optik der disperg. Medien,' p. 192 (Uppsala, 1920).

Debye's dispersion formula[*] for hydrogen is based definitely on Bohr's model; the methods used have been developed further by Sommerfeld.[†] Kirn, in the paper already quoted, calculates values of the refractive index from Debye's formula for comparison with his experimental results; these values, as given by Kirn, are shown in Column 8 of Table I. In this case the calculated values fall short of the observed, and increasingly so towards the shorter wave-lengths.

It will be seen that these various formulæ have analogous consequences as regards the scattering of light.

Scattering of Light by Hydrogen.

The quantity determined in Lord Rayleigh's experiment is the ratio of the intensities of the two polarised components of the scattered light when the primary beam is unpolarised; the revised value of this ratio for hydrogen is given as 0·0383.[‡] The corresponding formula on the present theory is (13), and we can calculate the value of this over the range of wave-lengths for which the dispersion formula is valid. Before doing so we may review briefly the other dispersion formulæ which were examined in the previous section.

Cuthbertson's formula, interpreted as meaning an isotropic molecule, gives zero value for the ratio (13), and this corresponds with the large defect of the calculated values of the dispersion in the short wave-lengths.

Debye's formula also falls short in the ultra-violet, as we saw in Table I. M. Born, in the papers quoted above, has used Debye's dispersion formula in calculating the ratio of intensities for the scattered light. His method is different from that used here; further his calculations give the ratio as negligibly small, apparently of the order of 10^{-5}.

On the other hand, Lundblad's formula gives a dispersion curve which rises too steeply in the ultra-violet, and so indicates a resonance wave-length which is too large. Since his dispersion formula is of the same type as (18) and (19) we can calculate the ratio of the intensities of the scattered light from (13); the value ranges from 0·1297 at the wave-length 5462 to 0·2175 at the wave-length 1854, and is therefore three or four times as large as the experimental value.

In the calculations for the new formulæ (18) and (19), we have the values of the sub-indices n_1 and n_2, and so the value of the ratio (13) can readily be found over the whole range. Although the expression (13) would require further modification near the resonance wave-lengths, just as in the corresponding dispersion formula, it is of interest to carry the calculations as far as the lines of the doublet in each case. Since $n^2-1=\frac{1}{3}\{n_1^2-1+2(n_2^2-1)\}$, it follows from (13) that the limiting values of the ratio are $\frac{1}{2}$ and $\frac{2}{9}$ at the lines of the doublet. The calculated values for some wave-lengths are shown in the second and third columns of Table II; the wave-lengths shorter than 1854 are the doublet lines given by (21) and (22).

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[*] P. Debye, 'Sitzungsber. d. Münch. Akad.,' p. 1 (1914).

[†] A. Sommerfeld, 'Ann. der Phys.,' vol. 53, p. 497 (1917).

[‡] Lord Rayleigh, 'Roy. Soc. Proc.,' A, vol. 97, p. 449 (1920). In Lord Rayleigh's earlier work ('Proceedings,' 1918) the value 0·017 was given for hydrogen; and the same value is adopted by J. Cabannes (Ann. de Physique, 15, 1921) to express his experimental results, which ranged between 0·01 and 0·02.

Table II.—Ratio of Intensities for Hydrogen.

λ.	(18).	(19).
5462	0·0326	0·0353
4359	0·0337	0·0361
3342	0·0358	0·0376
2753	0·0384	0·0394
1854	0·0517	0·0478
984	0·5000	—
924	—	0·2222
781	0·2222	—
699	—	0·5000

The difference between the values for the prolate and oblate types is of interest: the values from (18) begin by being less than those from (19), but they increase more rapidly owing to the larger maximum occurring at the longer wave-length of the doublet.

In comparing these results with Lord Rayleigh's value of 0·0383, it should be remembered that the latter is subject to experimental error (estimated at not more than 5 per cent.); further, the primary light was not monochromatic but was the radiation from an arc light. On the other hand, the calculated values involve possible experimental errors in the dispersion; and in addition the constants of the dispersion formula were not adjusted to give the best average agreement. It may be claimed that there is substantial agreement between the calculated values and experiment; but one could not decide between the alternative dispersion formulæ without more elaborate numerical analysis, and it would also probably need further experimental results. It may be that experiments with fluid or other media may offer less difficulties than in the case of gases to finding the variation of the ratio of intensities with wave-length; and a corresponding extension of the present analysis may be possible, connecting this variation with the dispersion formula of the medium.